PERTH MODERN SCHOOL
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**Semester One Examination, 2022**

**Question/Answer booklet**

# MATHEMATICS SPECIALIST UNIT 1

## Section Two: Calculator-assumed

**SOLUTIONS**

Your name _____

Teacher's name _____

## Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

## Materials required/recommended for this section

***To be provided by the supervisor***

This Question/Answer booklet
Formula sheet (retained from Section One)

***To be provided by the candidate***

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, approved for use in this examination Important note to candidates

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 6 | 6 | 50 | 47 | 35 |
| Section Two: Calculator-assumed | 11 | 11 | 100 | 93 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

**Section Two: Calculator-assumed** **65% (93 Marks)**

This section has **eleven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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**Question 7** **(8 marks)**

(a) Determine the number of ways that 13 persons can be seated in a row of 6 seats. (2 marks)

| **Solution** |
|---|
| ${}^{13}P_6 = 1235520$ |
| **Specific behaviours** |
| ✓ Uses combination ${}^{13}P_6$<br>✓ Determines the correct answer |

(b) A five-digit number divisible by 3 is to be formed using digits 0, 1, 2, 3, 4 and 5 without repetition. Determine the total number of ways this can be done. (3 marks)

| **Solution** |
|---|
| The set of number are {5, 4, 3, 2, 1} and {5, 4, 2, 1, 0}.<br><br>Using {5, 4, 3, 2, 1}: $5! = 120$<br><br>Using {5, 4, 2, 1, 0}: $4 \times 4! = 96$<br><br>Total is $120 + 96 = 216$ |
| **Specific behaviours** |
| ✓ Determines 120 for {5, 4, 3, 2, 1}<br>✓ Determines 96 for {5, 4, 2, 1, 0}<br>✓ Determines the correct total |

(c) There are 10 points in a plane, out of which 4 are collinear. Determine the number of triangles made by these 10 points. (3 marks)

| **Solution** |
|---|
| ${}^{10}C_3 - {}^{4}C_3 = 116$ |
| **Specific behaviours** |
| ✓ Uses ${}^{10}C_3$<br>✓ Excludes ${}^{4}C_3$<br>✓ Determines the correct total |

**Question 8** **(6 marks)**

(a) In a group of 20 people, friendship is mutual. Use Pigeonhole principle to show that there exist two people who have the same number of friends. (3 marks)

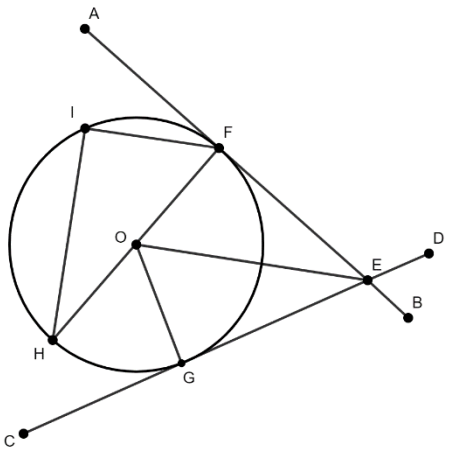
| **Solution** |
|---|
| Each person can have 0 to 19 friends. ✓<br>But if someone has 0 friends, then no one can have 19 friends and similarly you cannot have 19 friends and no friends. ✓<br>So, there are only 19 options for the number of friends and hence there exist two people who have the same number of friends. ✓ |
| **Specific behaviours** |
| ✓ See ticks above |

(b) A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white marbles. Determine the minimum number of marbles we must choose randomly from the box to ensure that we get 9 marbles of same colour. (3 marks)

| **Solution** |
|---|
| Among blue, yellow, and white which has more than 9 marbles each, we need to pick $8 \times 3 + 1 = 25$. ✓<br>we can get all the red and green balls before the above 25, therefore we need to add 6 red + 8 green as well. ✓<br>Hence, $6 + 8 + 25 = 39$ ✓ |
| **Specific behaviours** |
| ✓ Uses pigeonhole principal to conclude 25 needed from blue, yellow, and white<br>✓ Adds 6 red and 8 green to the total<br>✓ Determines the correct total |

**Question 9** **(9 marks)**

The diagram below (not to scale) showing a circle with centre at O and diameter HF. Point I lies on the circumference. AB and CD are tangents to the circle at Points F and G respectively. Point E is the intersection of AB and CD. Given $\angle IHF = 31°$ and $\angle DEB = 78°$, determine the size of each angle in the table, giving reason(s).




| | Size | Reason(s) |
|---|---|---|
| $\angle HIF$ | $90°$ | Angle in a semicircle |
| $\angle AFI$ | $31°$ | Alternate segment theorem |
| $\angle FOG$ | $102°$ | Tangent and radius are perpendicular<br>Vertically opposite angles<br>Angle sum of a quadrilateral |
| $\angle OEF$ | $39°$ | Two tangents from the same point<br>or<br>Congruent triangles (RHS/SSS/SAS) |

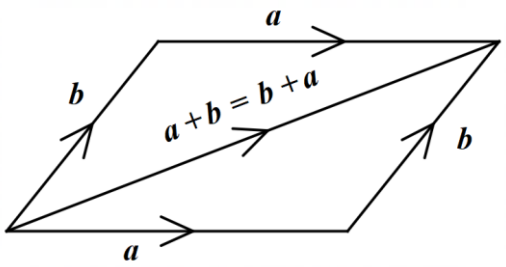
| **Solution** |
|---|
| See table. |
| **Specific behaviours** |
| ✓✓ states correct size of $\angle HIF$ with reason |
| ✓✓ states correct size of $\angle AFI$ with reason |
| ✓ states correct size of $\angle FOG$ |
| ✓ Gives the reason of tangent-radius |
| ✓ Gives at least one other reasons |
| ✓✓ states correct size of $\angle OEF$ with reason |

**Question 10** **(11 marks)**

(a) Draw two non-zero, non-parallel vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$.
Show with a diagram that

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$ (2 marks)

**Solution**




**Specific behaviours**

✓ Correct diagram for addition of vectors (head-to-tail with arrows)
✓ Correct diagram for same resultant vectors

(b) Two vectors are given as follows: $\boldsymbol{a} = 7\boldsymbol{i} + y\boldsymbol{j}$, and $\boldsymbol{b} = 18\boldsymbol{i} + 4\boldsymbol{j}$

(i) Find a value for $y$ such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel. (3 marks)

**Solution**

$$\begin{bmatrix} 7 \\ y \end{bmatrix} = k \begin{bmatrix} 18 \\ 4 \end{bmatrix}$$

$$k = \frac{7}{18}$$

$$y = \frac{7}{18} \times 4 = \frac{14}{9}$$

**Specific behaviours**

✓equation from $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel
✓ solves for constant
✓correct value for y

(ii) Find a value for $y$ such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. (3 marks)

**Solution**

$$\begin{bmatrix} 7 \\ y \end{bmatrix} . \begin{bmatrix} 18 \\ 4 \end{bmatrix} = 126 + 4y = 0$$

$$y = -\frac{63}{2}$$

**Specific behaviours**

✓equation from $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular
✓ Equation for y
✓correct value for y

(iii) Find the value(s) of $y$ if the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 60°. (3 marks)

| **Solution** |
| --- |
| $\text{solve}\left(\text{angle}\left(\begin{bmatrix}7\\y\end{bmatrix},\begin{bmatrix}18\\4\end{bmatrix}\right)=60,y\right)$<br>$\{y=-7.631452616, y=22.24014827\}$ |
| **Specific behaviours** |
| ✓✓ equation from angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 60°<br>✓correct value for y |

**Question 11** **(8 marks)**

(a) Prove that $7.\overline{53}$ is a rational number. (4 marks)

| **Solution** |
|---|
| Let $x = 7.\overline{53}$<br>$100x = 753.\overline{53}$<br>$99x = 746$<br>$x = \frac{746}{99}$<br><br>Therefore, $7.\overline{53}$ is a rational number. |
| **Specific behaviours** |
| ✓ writes equation for $100 \times 7.\overline{53}$<br>✓ eliminates recurring part<br>✓ ✓ express $7.5\overline{53}$ as a fraction |

(b) Let $p$ be an irrational number and $q$ a rational number. Use the method of proof by contradiction to prove that $p - q$ is irrational. (4 marks)

| **Solution** |
|---|
| Assume that $p$ is irrational and $q$ is rational, but that $p - q$ is rational.<br>Then $q = \frac{a}{b}$ and $p - q = \frac{c}{d}$ for some integers a, b, c and d where $b \neq 0, d \neq 0$ .<br>Now $p = q + (p - q)$<br>$= \frac{a}{b} + \frac{c}{d}$<br>$= \frac{ad + bc}{bd}$<br>which is rational since $ad + bc$ and $bd$ are integers. This contradicts the assumption that $p$ is irrational. Hence, $p - q$ must be irrational. |
| **Specific behaviours** |
| ✓ assumes that $p - q$ is rational<br>✓ writes **both** $q$ and $p - q$ as a ratio of integers<br>✓ calculates $q + (p - q)$ as a single fraction and hence rational<br>✓ notes contradiction and concludes that $p - q$ is irrational |

**Question 12** **(8 marks)**

5 letter "words" are chosen using the letters from the word DISCOVERY where each letter can only be used once.

(a) How many <u>different</u> words are possible? (2 marks)

| **Solution** |
|---|
| 9 x 8 x 7 x 6 x 5 = 15120 |
| **Specific behaviours** |
| ✓ ✓correct number |

(b) How many different words are possible which have exactly 2 vowels? (2 marks)

| **Solution** |
|---|
| ${}^3C_2 \times {}^6C_3 = 60$ different combinations of letters<br>$60 \times 5! = 7200$ |
| **Specific behaviours** |
| ✓ Worked out 60 combinations<br>✓ Multiplied combinations by 5! to get final answer |

(c) How many different words are possible that have the letters E, R and Y next to each other (in any order)? (2 marks)

| **Solution** |
|---|
| ERY can be ordered 3! = 6 different ways<br>6 x 5 x 6 x 3 = 540 |
| **Specific Behaviour** |
| ✓ Worked out 6 different ways that ERY are ordered<br>✓ Multiply for 90 to get final answer |

(d) How many different words start with a D and end in a Y? (2 marks)

| **Solution** |
|---|
| 1 x 7 x 6 x 5 x 1 = 210 |
| **Specific behaviours** |
| ✓ Fixes D and Y<br>✓ correct number |

**Question 13** **(7 marks)**

(a) Using ${}^{n}\mathbf{C}_{r} = \frac{n!}{r!(n-r)!}$, prove that ${}^{n}\mathbf{C}_{k} = {}^{n-1}\mathbf{C}_{k} + {}^{n-1}\mathbf{C}_{k-1}$ (4 marks)

| Solution |
|---|
| ${}^{n-1}C_k + {}^{n-1}C_{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$<br>$= \frac{(n-k)(n-1)!}{k!(n-k)!} + \frac{k(n-1)!}{k!(n-k)!}$<br>$= \frac{n(n-1)!-k(n-1)!+k(n-1)!}{k!(n-k)!}$<br>$= \frac{n(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = {}^{n}C_k$<br>Therefore ${}^{n}C_k = {}^{n-1}C_k + {}^{n-1}C_{k-1}$ |
| **Marking key/Mathematical behaviours** |
| ✓ Gives ${}^{n-1}C_k$ as $\frac{(n-1)!}{k!(n-k-1)!}$ and Simplifies ${}^{n-1}C_{k-1}$ as $\frac{(n-1)!}{(k-1)!(n-k)!}$<br>✓ Converts correctly to make common denominator of $k!\,(n-k)!$<br>✓ Adds together and simplifies to $\frac{n!}{k!(n-k)!}$<br>✓ Final statement proving initial proof |

(b) Find the positive integer k if ${}^{k}C_{k-4} = 2 \times {}^{k-1}C_{4}$ (3 marks)

| Solution |
|---|
| ${}^{k}C_{k-4} = \frac{k!}{(k-4)!(k-k+4)!} = \frac{k!}{(k-4)!4!}$<br>${}^{k-1}C_{4} = \frac{(k-1)!}{4!(k-1-4)!} = \frac{(k-1)!}{(k-5)!4!}$<br>$\frac{k!}{(k-4)!4!} = 2 \times \frac{(k-1)!}{(k-5)!4!}$<br>$\frac{k!}{(k-4)!4!} = 2 \times \frac{(k-4)k!}{k(k-4)!4!}$<br>$1 = 2 \times \frac{k-4}{k}$ $k = 2k - 8$ $k = 8$ |
| **Marking key/Mathematical behaviours** |
| ✓ Gives ${}^{k}C_{k-4}$ as $\frac{k!}{(k-4)!4!}$ and Simplifies ${}^{k-1}C_{4}$ as $\frac{(k-1)!}{(k-5)!4!}$<br>✓ Rearranges to get correct equation involving k<br>✓ Solves equation for k |

**Question 14** **(9 marks)**

(a) Points $A, B, C$ and $D$ lie in order on the circumference of the circle with centre $O$ so that $AB = 24.0$ cm, $BC = 16.1$ cm, and $AC$ and $BD$ are diameters. Determine, with brief reasons and to the nearest degree, the sizes of $\angle ACB$, $\angle ADB$, $\angle AOB$ and $\angle ABD$. (5 marks)

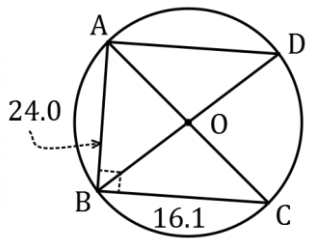
**Solution**

$$\angle ACB = \tan^{-1}\left(\frac{24.0}{16.1}\right) = 56°$$

$$\angle ADB = \angle ACB = 56° \text{ (angle on same arc)}$$

$$\angle AOB = 2\angle ACB = 112° \text{ (angle at centre)}$$

$$\angle ABD = 90° - 56° = 34° \text{ (angle in semicircle)}$$




**Specific behaviours**

✓ ✓calculates $\angle ACB$
✓✓✓ calculates each angle, with reasoning
Max –1 for insufficient reasons

(b) Points $P, Q$ and $R$ lie on the circumference of a circle of radius 11.7 cm, so that $PR = 10.3$ cm and $QR = 20.8$ cm. Prove by contradiction that the midpoint of chord $PQ$ is not the centre of the circle. (4 marks)

**Solution**

Assume that the midpoint of $PQ$ is the centre of the circle.

Hence $PQ$ is a diameter of the circle and the angle in a semicircle theorem implies that $\Delta PQR$ must be right angled at $R$.

Using Pythagoras' theorem, the length of diameter $PQ$ is given by

$$\begin{aligned} PQ &= \sqrt{PR^2 + QR^2} \\ &= \sqrt{10.3^2 + 20.8^2} \\ &= 23.21 \text{ cm} \end{aligned}$$

Hence the radius of the circle is $23.21 \div 2 = 11.605$ cm.

This result contradicts the fact that the radius of the circle is $11.7$ cm and so our assumption is wrong and thus the midpoint of chord $PQ$ is not the centre of the circle.

**Specific behaviours**

✓ states assumption
✓ uses assumption to imply that $\Delta PQR$ is right angled
✓ calculates diameter of circle
✓ uses contradiction to complete proof

**Question 15** **(8 marks)**

Relative to boat $O$ at anchor in a lake, four buoys $A, B, C$ and $D$ have the following position vectors (with distances in metres):

$$\overrightarrow{OA} = \begin{pmatrix}-380\\-420\end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix}-12\\342\end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix}420\\550\end{pmatrix}, \quad \overrightarrow{OD} = \begin{pmatrix}268\\-108\end{pmatrix}.$$

(a) Prove that the quadrilateral with vertices $ABCD$ is a trapezium, but not a parallelogram. (5 marks)

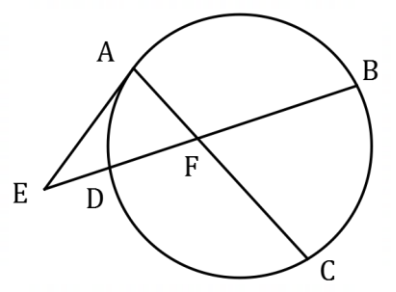
| **Solution** |
|---|
| Displacement vectors for all four sides are<br>$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (368, 762)$<br>$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = (152, 658)$<br>$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (432, 208)$<br>$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (648, 312)$<br>Using **i**-coefficients, $\frac{648}{432}\overrightarrow{BC} = (648, 312) = \overrightarrow{AD}$ and hence $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$.<br>Also, $\frac{368}{152}\overrightarrow{DC} = (368, \frac{53298}{19}) \approx (368, 2805) \neq \overrightarrow{AB}$ and hence $\overrightarrow{DC}$ is not parallel to $\overrightarrow{AB}$.<br>Hence $ABCD$ has just one pair of parallel sides and thus is a trapezium but not a parallelogram. |
| **Specific behaviours** |
| ✓ calculates correct displacement vectors for at least one side<br>✓ calculates correct displacement vectors for all sides<br>✓ clearly shows $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$<br>✓ clearly shows $\overrightarrow{DC}$ is not parallel to $\overrightarrow{AB}$<br>✓ uses results to justify $ABCD$ is a trapezium but not a parallelogram |

(b) Boat $X$ motors directly from $D$ to $B$ with a constant velocity in 3 minutes and 20 seconds. Determine the velocity in component form, and hence the speed, of boat $X$. (3 marks)

| **Solution** |
|---|
| Displacement vector $\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD} = (-280, 450)$ m.<br>Hence velocity vector $\mathbf{v}_{DB} = \overrightarrow{DB} \div 200 = (-1.4, 2.25)$ m/s.<br>Hence speed $= \lvert\mathbf{v}_{DB}\rvert = 2.65$ m/s. |
| **Specific behaviours** |
| ✓ displacement vector<br>✓ velocity vector<br>✓ speed, with units. |

**Question 16** **(9 marks)**

(a) In the diagram (not to scale), $EA$ is a tangent to the circle at $A$. Secant $BE$ cuts chord $AC$ at $F$, and the circle at $D$.

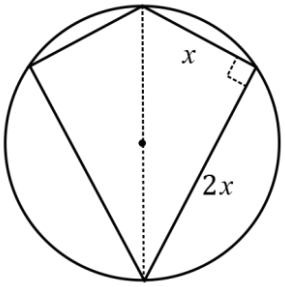



$DE = 5$ cm, $DB = 40$ cm, $AE = FE$, and $CF$ is 1 cm longer than $AE$.

Determine the length of $AF$. (4 marks)

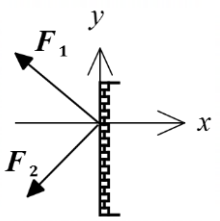
| Solution |
|---|
| Secant-tangent property:<br>$BE \times DE = EA^2$<br>$(40 + 5) \times 5 = EA^2$<br>$EA = 15$ cm<br>Intersecting chords property:<br>$CF \times AF = BF \times DF$<br>$(15 + 1) \times AF = \big(40 - (15 - 5)\big)(15 - 5)$<br>$16AF = 30 \times 10$<br>$AF = 18.75$ cm |
| **Specific behaviours** |
| ✓ correct use of secant-tangent property<br>✓ length $EA$<br>✓ correct use of intersecting chords property<br>✓ length $AF$ |

(b) The vertices of a kite lie on the circumference of a circle. Each longer side of the kite is twice the length of the adjacent shorter side. If the area of the kite is $18$ cm$^2$, determine the radius of the circle. (5 marks)

| Solution | |
|---|---|
| Since vertices of a kite lie on the circumference of a circle, it is a cyclic quadrilateral. The sum of opposite angles is $180°$. The two triangles are congruent since the corresponding sides are equal.<br>∴The two triangles of kite are right triangles.<br>∴ Hypotenuse of the triangle is the diameter of the circle. (Angle in semicircle is 90°)<br>Hence: $A = \frac{1}{2}(2x)(x) \times 2$<br>$2x^2 = 18$<br>$x = 3$<br>Hence<br>$(2r)^2 = 6^2 + 3^2$<br>$r = \frac{3\sqrt{5}}{2} \approx 3.35$ cm |  |
| **Specific behaviours** | |
| ✓Uses kite as a cyclic quadrilateral to show triangles are congruent<br>✓ uses angles in semicircle property to show that hypotenuse is a diameter<br>✓ forms area equation<br>✓ solves equation<br>✓ correct radius | |

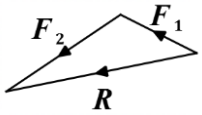
**Question 17** **(10 marks)**

The diagram at right, not to scale, shows forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting in the same vertical plane on a small hook fixed to a vertical wall. $\mathbf{F}_1$ has magnitude $147$ N and acts at an angle of elevation of $22°$ and $\mathbf{F}_2$ has magnitude $195$ N and acts at an angle of depression of $42°$.




The resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ is $\mathbf{R}$.

(a) Sketch a triangle to show the relationship between $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{R}$. (2 marks)

| **Solution** |
|---|
|   |
| **Specific behaviours** |
| ✓ ✓nose-to-tails force vectors and completes triangle with resultant |

(b) Determine, with reasoning, the magnitude of $\mathbf{R}$ and the acute angle it makes with the wall. (5 marks)

| **Solution** |
|---|
| Angle in triangle between forces is $180° - 22° - 42° = 116°$. <br><br> $\|\mathbf{R}\| = 147^2 + 195^2 - 2(147)(195)\cos 116°$ <br> $= 291.15 \approx 291$ N <br><br> Let $\theta$ be the angle between $\mathbf{F}_1$ and $\mathbf{R}$: <br> $\frac{\sin\theta}{195} = \frac{\sin 116°}{291.15}$ <br> $\theta = 37.0°$ <br><br> Hence acute angle with wall is $90° + 22° - 37° = 75°$. |
| **Specific behaviours** |
| ✓ correct angle between forces (*shown here or in (a)*) <br> ✓ expression using cosine rule with magnitude <br> ✓ calculates magnitude <br> ✓ expression using sine rule with angle <br> ✓ calculates angle with horizontal |

(c) The wall exerts a force on the hook of equal magnitude to $\mathbf{R}$ but in the opposite direction. Express this force using unit vectors $\mathbf{i}$ and $\mathbf{j}$. (3 marks)

| **Solution** |
|---|
| Angle $\mathbf{R}$ makes with $x$-axis is $-90° - 75° = -165°$. <br><br> $\mathbf{R} = 291.15(\cos(-165°), \sin(-165°))$ <br> $= -281.2\mathbf{i} - 75.4\mathbf{j}$ <br><br> Hence force exerted by wall is $281.2\mathbf{i} + 75.4\mathbf{j}$. |
| **Specific behaviours** |
| ✓ indicates angle of resultant with $x$-axis, or similar <br> ✓ converts into component form <br> ✓ correctly reverses direction |

Supplementary page

Question number: _____